Phase transition in an asymmetric generalization of the zero-temperature Glauber model

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An asymmetric generalization of the zero-temperature Glauber model on a lattice is introduced. The dynamics of the particle-density and especially the large-time behavior of the system is studied. It is shown that the system exhibits two kinds of phase transitions, a static one and a dynamic one.

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I. INTRODUCTION

In recent years, reaction–diffusion systems have been studied by many people. As mean-field techniques, generally, do not give correct results for low-dimensional systems, people are motivated to study exactly solvable stochastic models in low dimensions. Moreover, solving one-dimensional systems should in principle be easier. Exact results for some models on a one–dimensional lattice have been obtained, for example, in Refs. [1–11]. Different methods have been used to study these models, including analytical and asymptotic methods, mean-field methods, and large-scale numerical methods.

Some interesting problems in nonequilibrium systems are nonequilibrium phase transitions described by phenomenological rate equations, and the way the system relaxes to its steady state. Kinetic generalizations of the Ising model, for example, the Glauber model or the Kawasaki model, are such phenomenological models and have been studied extensively [12–17]. Combination of the Glauber and the Kawasaki dynamics has been also considered [18–20].

In this paper, we want to study an asymmetric generalization of the zero-temperature Glauber model on a lattice with boundaries. There are also sources (or sinks) of particles at the end points of the lattice. We study the dynamics of the particle density, and especially the large-time behavior of the system. In the thermodynamic limit, the system shows two kinds of phase transitions. One of these is a static phase transition, the other a dynamic one. The static phase transition is controlled by the reaction rates, and is a discontinuous change of the behavior of the derivative of the stationary particle density at the end points, with respect to the reaction rates. The dynamic phase transition is controlled by the injection and extraction rates of the particles at the end points, and is a discontinuous change of the relaxation time towards the stationary configuration.

II. ASYMMETRIC GLAUBER MODEL AT ZERO TEMPERATURE

In the ordinary Glauber model, the interaction is between three neighboring sites. Spin flip brings the system to equilibrium with a heat bath at temperature T. A spin is flipped with the rate $\mu := 1 - \tanh J/kT$, if the spin of both of its neighboring sites are the same as itself, and is flipped with the rate $\lambda := 1 + \tanh J/kT$ if the spin of both of its neighboring sites are opposite to it. At domain boundaries, the spins are flipped with unit rate. So the interactions can be written as

$$AAA \to A\emptyset\emptyset A \quad \text{and} \quad \emptyset\emptyset\emptyset \to \emptyset A\emptyset \quad \mu,$$

$$A\emptyset A \to AAA \quad \text{and} \quad \emptyset A\emptyset \to \emptyset\emptyset\emptyset \quad \lambda,$$

$$AA\emptyset \rightleftharpoons A\emptyset\emptyset \quad \text{and} \quad \emptyset\emptyset A \rightleftharpoons \emptyset AA \quad 1,$$

where spin up and spin down are denoted by A and $\emptyset$. One can interpret an up spin as a particle, and a down spin as a hole. At zero temperature, the Glauber dynamics is effectively a two-site interaction [21]:

$$A\emptyset \to \begin{cases} AA \\ \emptyset\emptyset, \end{cases} \tag{1}$$

$$\emptyset A \to \begin{cases} AA \\ \emptyset\emptyset, \end{cases} \tag{2}$$

where all the above processes occur with the same rate.

One can consider the following interactions, as an asymmetric generalization of the zero-temperature Glauber model.

$$A\emptyset \to \begin{cases} AA & u \\ \emptyset\emptyset & v, \end{cases} \tag{3}$$

$$\emptyset A \to \begin{cases} AA & v \\ \emptyset\emptyset & u. \end{cases} \tag{4}$$

If $u \neq v$, the above system has left-right asymmetry. The above system on an infinite lattice has been investigated in Ref. [22], where its n-point functions, its equilibrium states, and its relaxation towards these states are studied. It can be easily shown that the time evolution equation for the average densities of the system with the above interactions are the same as that of a system with the following interactions, where diffusion is also present:

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$$A\emptyset \rightarrow \begin{cases} \emptyset A & \Lambda \\ AA & u-\Lambda \\ \emptyset\emptyset & v-\Lambda, \end{cases} \tag{5}$$

$$\emptyset A \rightarrow \begin{cases} A\emptyset & \Lambda' \\ AA & v-\Lambda' \\ \emptyset\emptyset & u-\Lambda'. \end{cases} \tag{6}$$

Consider a lattice with L sites and an asymmetric zero-temperature Glauber dynamics as the interaction. The rates of injection and extraction of particles in the first site (final site) are a and a' (b and b'), respectively. The time evolution equations for the average densities are then

$$\langle \dot{n}_k \rangle = -(u+v)\langle n_k \rangle + u\langle n_{k-1} \rangle + v\langle n_{k+1} \rangle \quad \text{for} \quad k \neq 1,L,$$

$$\langle \dot{n}_1 \rangle = a - (a+a'+v)\langle n_1 \rangle + v\langle n_2 \rangle, \tag{7}$$

$$\langle \dot{n}_L \rangle = b - (b+b'+u)\langle n_L \rangle + u\langle n_{L-1} \rangle.$$

First, let us calculate the profile of average densities at large times. At large times, the system goes to its stationary state, and the time derivatives of the left–hand sides vanish. One can see then that the solution to the above system is

$$\langle n_k(\infty) \rangle = \alpha + \beta \left(\frac{u}{v} \right)^k. \tag{8}$$

Putting this in Eq. (7), one can easily find α and β. In the thermodynamic limit ($L \rightarrow \infty$), the solution becomes

$$\langle n_k(\infty) \rangle = \frac{a}{a+a'} + \frac{ba'-ab'}{(a+a')(b+b'+u-v)} \left(\frac{u}{v} \right)^{k-L},$$

$$u > v,$$

$$\langle n_k(\infty) \rangle = \frac{b}{b+b'} + \frac{ab'-a'b}{(b+b')(a+a'+v-u)} \left(\frac{u}{v} \right)^{k-1},$$

$$u < v. \tag{9}$$

It is seen that the density profile is flat at the left end ($k \ll L$) for $u > v$ and its value is independent of the reaction rates. But as v exceeds u, the density profile acquires a finite slope, proportional to $\ln(u/v)$. This is the static phase transition previously mentioned.

Now return to the dynamics of the system. Let us write the homogeneous part of Eq. (7) as

$$\langle \dot{n}_k \rangle = h_k^l \langle n_l \rangle, \tag{10}$$

and find the eigenvalues and eigenvectors of the operator h. One finds

$$Ex_k = -(u+v)x_k + ux_{k-1} + vx_{k+1}, \quad k \neq 1,L,$$

$$Ex_1 = -(a+a'+v)x_1 + vx_2,$$

$$Ex_L = -(b+b'+u)x_L + ux_{L-1}, \tag{11}$$

where the eigenvalue and eigenvector have been denoted with E and x, respectively. The solution to these equations is

$$x_k = \alpha z_1^k + \beta z_2^k, \tag{12}$$

where z_i's satisfy

$$E = -(u+v) + vz + \frac{u}{z}, \tag{13}$$

and

$$v(\alpha z_1^2 + \beta z_2^2) - (E+a+a'+v)(\alpha z_1 + \beta z_2) = 0,$$

$$u(\alpha z_1^{L-1} + \beta z_2^{L-1}) - (E+b+b'+u)(\alpha z_1^L + \beta z_2^L) = 0. \tag{14}$$

To have nonzero solutions for x, these last two equations should be dependent, the criterion of which is

$$(u+z_1\delta a)(vz_2^{L+1} + z_2^L \delta b) - (u+z_2\delta a)(vz_1^{L+1} + z_1^L \delta b) = 0, \tag{15}$$

where Eq. (13) has been used to eliminate E, and $\delta a := a + a' - u$ and $\delta b := b + b' - v$. Defining

$$Z_i := z_i \sqrt{\frac{v}{u}},$$

$$A := \frac{\delta a}{\sqrt{uv}}, \tag{16}$$

$$B := \frac{\delta b}{\sqrt{uv}},$$

Eq. (15) is simplified to

$$Z^{-(L+1)}(1+AZ)(1+BZ) - Z^{L+1}(1+AZ^{-1})(1+BZ^{-1}) = 0. \tag{17}$$

The eigenvalue E satisfies

$$E = -(u+v) + \sqrt{uv}(Z + Z^{-1}). \tag{18}$$

Two obvious solutions of the Eq. (17) are $Z = \pm 1$. But these generally don't correspond to eigenvalues and eigenvectors. In fact for these solutions, Z and Z^{-1} are the same, so that Eq. (12) should be modified to

$$x_k = (\alpha + \beta k)(\pm 1)^k, \tag{19}$$

and it is not difficult to see that these do not fulfill the boundary conditions unless $\alpha = \beta = 0$. Equation (18) can be written as a polynomial equation of order $2(L+1)$, and hence has $2L$ more roots in addition to $Z = \pm 1$. For these $2L$ roots, if Z is a root Z^{-1} is another root, and these two correspond to

one eigenvalue and one eigenvector. So the $L\times L$ matrix h does have L eigenvalues and eigenvectors.

For $A=B=0$, Eq. (17) is very simple and its nontrivial solutions are

$$Z_{(s)}=e^{i\pi s/(L+1)}, \quad 1\leqslant s\leqslant L, \tag{20}$$

and their inverses. All of these are phases and the real part of the corresponding eigenvalues satisfy

$$\mathrm{Re}(E)\leqslant -(u+v)+2\sqrt{uv}\cos\left(\frac{\pi}{L+1}\right)<0. \tag{21}$$

The maximum of the real part of the eigenvalues determines the relaxation time toward the stationary average-density profile. That is

$$\tau=\left[u+v-2\sqrt{uv}\cos\left(\frac{\pi}{L+1}\right)\right]^{-1}. \tag{22}$$

In the limit $L\to\infty$, this is simplified to

$$\tau=(u+v-2\sqrt{uv})^{-1}. \tag{23}$$

The general solution to Eq. (7) is seen to be

$$\langle n_k(t)\rangle=\sum_{s,m}\frac{2}{L+1}\exp[E_{(s)}t]\langle n_k(0)\rangle \times\left(\frac{u}{v}\right)^{(k-m)/2}\sin\left(\frac{\pi sk}{L+1}\right)\sin\left(\frac{\pi sm}{L+1}\right). \tag{24}$$

Now consider the general case. Equation (17) can be written as

$$G(Z):=F(Z)-F(Z^{-1})=0, \tag{25}$$

where

$$F(Z):=Z^{-(L+1)}(1+AZ)(1+BZ). \tag{26}$$

If Z is a phase, it satisfies Eq. (17) provided $F(Z)$ is real. Consider the phase of $F(Z)$ for unimodular Z. We have

$$\phi[F(z)]=-(L+1)\phi(Z)+\phi(1+AZ)+\phi(1+BZ), \tag{27}$$

where ϕ denotes the phase of its argument. As the phase of Z is changed from 0 to 2π, the change of the phase of $F(z)$ is

$$\Delta\phi[F(Z)]=-2\pi(L+1)+\Delta\phi(1+AZ)+\Delta\phi(1+BZ). \tag{28}$$

For $|A|,|B|<1$, the phase changes of $(1+AZ)$ and $(1+BZ)$ are zero, as Z moves on the whole unit circle. So,

$$\Delta\phi[F(Z)]=-2\pi(L+1), \quad \text{for } |A|,|B|<1. \tag{29}$$

But this means that the phase of $F(Z)$ will be an integer multiple of π for at least $2(L+1)$ points on the unit circle. So all of the solutions of Eq. (17) are still phases, although they may be not uniformly spaced on the unit circle. Two of these are $Z=\pm 1$. The remaining $2L$ points correspond to L eigenvalues and eigenvectors for h. One concludes that for $|A|,|B|<1$,

$$\tau=(u+v-2\sqrt{uv}\cos\theta)^{-1}, \tag{30}$$

for some θ. Especially, at $L\to\infty$, the relaxation time is the same as the relaxation time for $A=B=0$, that is the same as Eq. (23).

If, for example, $|A|>1$, then the total phase change of $(1+AZ)$ is no longer zero. It is 2π. One may then lose two of the roots of the unit circle. Note that the mere fact that $\Delta\phi[F(Z)]=-2\pi L$ does not mean that there are just $2L$ solutions of Eq. (17) on the unit circle, since the phase of $F(Z)$ need not be monotonic. To find values of A and B, for them the number of the solutions of Eq. (17) on the unit circle is $2L$ or $2(L-1)$, consider the function $G(Z)$ at the points $Z=\pm 1$. Increasing $|A|$ or $|B|$, two of the roots on the unit circle tend to 1 or -1, and then move out of the unit circle and on the real line. At the point that this occurs, either $G'(1)$ or $G'(-1)$ become zero, as there will be multiple roots at ± 1. So the criterion for each change (losing 2 roots of the unit circle) is that either $G'(1)$ or $G'(-1)$ become zero. The curves in the AB plane, corresponding to these changes are

$$L(1+A)(1+B)+1-AB=0, \quad G'(1)=0, \tag{31}$$

and

$$L(1-A)(1-B)+1-AB=0, \quad G'(-1)=0. \tag{32}$$

These curves divide the plane into six regions:

(I) all of the solutions are phases.
(II) $2L$ phase solutions, 2 real negative solutions.
(III) $2(L-1)$ phase solutions, 4 real negative solutions.
(IV) $2L$ phase solutions, 2 real positive solutions.
(V) $2(L-1)$ phase solutions, 4 real positive solutions.
(VI) $2(L-1)$ phase solutions, 2 real negative solutions, 2 real positive solutions.

In the above, by real solutions is meant real solutions besides the trivial solutions ± 1. Note, however, that not all of this plane is physical. The physical region is that part of the plane, which corresponds to non-negative values for the injection and extraction rates. Returning to the definitions of A and B, Eq. (16), it is seen that

$$A_{\min}=-\sqrt{\frac{u}{v}}, \qquad B_{\min}=-\sqrt{\frac{v}{u}}. \tag{33}$$

The point $(A_{\min},B_{\min})$ itself is on the curve

$$A_{\min}B_{\min}=1. \tag{34}$$

One can see that, unless $u=v$, part of the physical region is in the region IV, where two of the solutions of Eq. (17) are

real positive (and of course the inverse of each other). This makes the maximum of the real part of E larger than $-(u+v)+2\sqrt{uv}$, and correspondingly, the relaxation time larger than Eq. (23).

In the thermodynamic limit $L\to\infty$, the regions of the AB plane are greatly simplified. In fact the curves corresponding to $G'(\pm 1)=0$ become

$$A=-1, \quad \text{or} \quad B=-1, \quad G'(1)=0, \tag{35}$$

and

$$A=1, \quad \text{or} \quad B=1, \quad G'(-1)=0. \tag{36}$$

In the case where either $|A|$ or $|B|$ are greater than 1, the real roots of Eq. (17) are simply

$$Z=-A,-A^{-1}, \quad |A|>1 \tag{37}$$

and

$$Z=-B,-B^{-1}, \quad |B|>1. \tag{38}$$

Then, if for example, A is negative and less than -1 and $A<B$, the maximum real part of E is $-(u+v)-\sqrt{uv}(A+A^{-1})$, and the relaxation time of the system is

$$\tau=[u+v+\sqrt{uv}(A+A^{-1})]^{-1}, \tag{39}$$

which is greater than Eq. (23). This is a phase transition that occurs at $A=-1$. For $A>-1$, the relaxation time is constant, Eq. (23). For $A<-1$, it is A dependent. The minimum of A is $-\sqrt{u/v}$, for which one of the eigenvalues of h becomes zero, and the relaxation time becomes infinite. The same effect is seen for $B<-1$. As mentioned before, for $u=v$ (the ordinary zero-temperature Glauber model) no part of the region IV is in the physical region, and this transition does not occur. The phase transition discussed here is the dynamical phase transition mentioned before.

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